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Dequantization of Real Convex Projective Manifolds

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Abstract. This paper is about the construction of compactification for the parameter space of convex projective structures on a fixed manifold. The parameter space is a semi-algebraic set, and the compactification is constructed by applying the Maslov dequantization to this set, constructing the so-called logarithmic limit set. The interpretation of boundary points is given by the "dequantization", in a suitable sense, of actions of the fundamental group of the manifold, on projective spaces.

1. Introduction

This paper is a survey of a work about the compactification of the parameter space of convex projective structures on an n-manifold. The complete work is split between the papers [**1**], [**2**], [**3**], and the reader is referenced there for complete proofs. Here we concentrate on giving the definitions and the main ideas.

In this paper we will work with a closed orientable n-manifold M whose universal covering is $\mathbb{R}^n$ and such that $\pi_1(M)$ is Gromov-hyperbolic. For example every closed orientable hyperbolic n-manifold satisfies these hypotheses. In low dimension, when $n = 2$ or 3, it is known that M has to be an hyperbolic manifold. We denote by $\mathcal{T}^c_{\mathbb{RP}^n}(M)$ the parameter space of marked convex projective structures on M. We construct a compactification $\mathcal{T}^c_{\mathbb{RP}^n}(M) \cup \partial\mathcal{T}^c_{\mathbb{RP}^n}(M)$ such that the action of the mapping class group of M extends continuously to an action on the boundary. During this construction we use some important results of Benoist about convex projective manifolds, see [**4**].

This construction generalizes the compactification of Teichmüller spaces, in the approach of Morgan and Shalen, see [**12**]. We extended their theory and their compactification construction, so that it can be used to compactify the spaces of convex projective structures. The construction of Morgan and Shalen already contained some elements that now are considered part of tropical geometry, but in our approach we make an explicit use of the tropical semifield and Maslov dequantization, and the present account of the work is all around the dequantization idea.

When a space is compactified, new points are added to it to form a boundary. These boundary points can be considered as points at infinity. If we apply the

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Maslov dequantization to a real semi-algebraic set, what we see in the limit object represents the behavior of the set near infinity. Hence the limit object, here called logarithmic limit set, can be glued to the semi-algebraic set in a natural way giving a compactification. More work is needed if we want to extend the action of a group to the compactification, see below.

If the space compactified is a parameter space, one would like to consider also the boundary as a parameter space. The objects parametrized by the boundary can be thought of as degenerate versions of the objects parametrized by the interior points. In tropical geometry, algebraic varieties degenerate, via the Maslov dequantization, to tropical varieties. In our case we work with real convex projective structures on a manifold M, and degenerate versions of such objects can be something like tropical projective structures on M. We give a definition of what should be a tropical projective structure, and we use these objects as an interpretation for the boundary points of our parameter space.

Interestingly enough, the boundary is constructed as the Maslov dequantization, or tropicalization, of the parameter space, and the boundary points are interpreted as tropicalizations of the interior points. Hence the Maslov dequantization appears in two a priori unrelated ways.

A brief summary of the following sections. We start with some linear algebra over semifields, as we want to define what is a projective space over a semifield, see section 2.

Then we presents some examples of projective spaces that are important for our work. The most important are the convex subsets of $\mathbb{RP}^n$, that are projective spaces over $\mathbb{R}_{\geq 0}$, and their tropical counterparts, that are, surprisingly enough, the Bruhat-Tits buildings with a structure of projective spaces over the tropical semifield, see section 3.

We show that it is possible to generalize the Hilbert metric, a projectively invariant distance defined naturally on the open convex subsets of $\mathbb{RP}^n$, to generic projective spaces over the tropical semifields, see section 4.

Finally we are ready to define the main objects of the paper, convex real projective structures and their tropical counterparts. We also define the length spectra associated with such structures, see section 5.

Then we need to construct and describe the parameter spaces of marked convex projective structures. First we need to introduce the variety of characters of representations of a finitely generated group in $SL_{n+1}(\mathbb{R})$. Such varieties are closed semi-algebraic sets, see section 6.

Using the varieties of characters we can describe the parameter space of marked convex projective structures on M. Such spaces are again closed semi-algebraic sets, see section 7.

The construction of compactification is then presented in a general way, for general closed semi-algebraic sets. The tool used is the logarithmic limit set and the Maslov dequantization, see section 8.

In the last section we use all the previously stated results to construct the compactification of the parameter space of marked convex projective structures on M, and we prove the theorem about the interpretation of the boundary points, see section 9.

2. Projective spaces over semifields

2.1. Semifields. A **semifield** is a quintuple $(S, +, \cdot, 0, 1)$, where S is a set, $+$ and $\cdot$ are associative and commutative operations $S \times S \mapsto S$ satisfying the distributivity law, $0, 1 \in S$ are, respectively, the neutral elements for $+$ and $\cdot$. Moreover we require that every element $a \in S^* = S \setminus \{0\}$ has a multiplicative inverse a^{-1}. Given an element $b \neq 0$ we can write $a/b = a \cdot b^{-1}$. Note that 0 is never invertible and $\forall s \in S, 0 \cdot s = 0$.

A semifield is a field if and only if every element has an additive inverse. If a semifield is not a field, it is a **zerosumfree semifield**, i.e. if $x + y = 0$, then $x = y = 0$.

For example, if $\mathbb{F} = (F, +, \cdot, 0, 1, \leq)$ is an ordered field, then

$$\mathbb{F}_{\geq 0} = (\{x \in F \mid x \geq 0\}, +, \cdot, 0, 1)$$

is a zerosumfree semifield. Semifields of the form $\mathbb{F}_{\geq 0}$ are **cancellative** semifields, i.e. if $a + b = c + b$ then $a = c$. Cancellative semifields behave very similarly to rings and fields.

On the other extreme there are **idempotent** semifields, where $\forall s \in S, s+s = s$. Clearly these semifields are never cancellative. It is possible to construct some examples of idempotent semifields starting from an abelian ordered group $(\Lambda, +, <)$. We add to it an extra element $-\infty$ with the property $\forall \lambda \in \Lambda, -\infty < \lambda$, and we define a zerosumfree semifield:

$$\mathbb{T}_\Lambda = (\Lambda \cup \{-\infty\}, \oplus, \odot, -\infty, 0)$$

with the tropical operations $\oplus, \odot$ defined as

$$a \oplus b = \max(a, b)$$

$$a \odot b = \begin{cases} a + b & \text{if } a, b \in \Lambda \\ -\infty & \text{if } a = -\infty \text{ or } b = -\infty \end{cases}$$

We will use the notation $1_\mathbb{T} = 0$, as the zero of the ordered group is the one of the semifield, and $0_\mathbb{T} = -\infty$. If $a \in \mathbb{T}_\Lambda$ and $a \neq 0_\mathbb{T}$, then $a \odot (-a) = 1_\mathbb{T}$. Hence $-a = a^{\odot -1}$, the **tropical inverse** of a. We will write $a \oslash b = a \odot b^{\odot -1} = a - b$. Semifields of the form $\mathbb{T}_\Lambda$ will be called **tropical semifields**. The semifield that in literature is called the tropical semifield is, in our notation, $\mathbb{T}_\mathbb{R}$.

There are two constructions relating ordered fields and idempotent semifields: Maslov dequantization and valuations. The two constructions are actually two different ways for seeing the same thing.

Given a number $t \in (0, 1)$, consider the function:

$$\mathbb{R}_{\geq 0} \ni z \mapsto \log_{\left(\frac{1}{t}\right)} z = \left(\frac{-1}{\log t}\right) \log z \in \mathbb{R} \cup \{-\infty\}$$

This function is bijective, with inverse $x \mapsto t^{-x}$, and it preserves the order $\leq$. The operations ('+' and '·') are transformed via conjugation in the following way:

$$x \oplus_t y = \log_{\left(\frac{1}{t}\right)}(t^{-x} + t^{-y})$$

$$x \odot_t y = \log_{\left(\frac{1}{t}\right)}(t^{-x} \cdot t^{-y}) = x + y$$

Hence every t induces a semifield structure on $\mathbb{R} \cup \{-\infty\}$, isomorphic to $\mathbb{R}_{\geq 0}$:

$$\mathbb{R}^t = (\mathbb{R} \cup \{-\infty\}, \oplus_t, \odot_t, -\infty, 0)$$

In the limit for t tending to zero we have:

$$\lim_{t\to 0^+} x \oplus_t y = \max(x, y)$$

The limit semifield is $\mathbb{T}_\mathbb{R}$, the tropical semifield. This construction is usually called **Maslov dequantization**.

The tropical semifields are the images of valuations. Let $\mathbb{F}$ be a field, Λ an ordered group, and $v : \mathbb{F} \mapsto \Lambda \cup \{+\infty\}$ a surjective valuation. Instead of using the valuation, we prefer the **tropicalization map**:

$$\tau : \mathbb{F} \ni z \mapsto -v(z) \in \mathbb{T} = \mathbb{T}_\Lambda = \Lambda \cup \{-\infty\}$$

The tropicalization map satisfies the properties of a norm: $\tau(z) = 0_\mathbb{T} \Leftrightarrow z = 0$, $\tau(zw) = \tau(z) \odot \tau(w)$, $\tau(z + w) \leq \tau(z) \oplus \tau(w)$. We will denote the valuation ring by $\mathcal{O} = \{z \in \mathbb{F} \mid \tau(z) \leq 1_\mathbb{T}\}$.

To relate the two constructions, consider the case when $\mathbb{F} = \mathbb{R}(t)$, the field of rational functions, taking the degree as valuation v, with tropicalization map τ. This field has a unique order such that, for every real number ε, $0 < t < \varepsilon$. Note that the valuation v respects this order. Every element $f \in \mathbb{R}(t)_{>0}$ corresponds to a function $f : (0, \varepsilon) \mapsto \mathbb{R}_{>0}$, and this function can be interpreted as a one parameter family of positive real numbers. By applying the Maslov dequantization to this family, the function becomes

$$\log_{\left(\frac{1}{t}\right)} f(t) = \left(\frac{-\log f(t)}{\log t}\right)$$

with $t \in (0, \varepsilon)$. It is easy to compute the limit

$$\lim_{t\to 0} \log_{\left(\frac{1}{t}\right)} f(t) = \tau(f)$$

2.2. Semimodules. Many interesting geometric objects are projective spaces over a semifield, and the maps preserving their geometric structure are projective maps. We will see some examples: polytopes and other convex subsets of $\mathbb{RP}^n$ are projective spaces over $\mathbb{R}_{\geq 0}$, the Bruhat-Tits buildings of SL_n are projective spaces over some tropical semifield.

DEFINITION 2.1. Given a semifield S, an S-**semimodule** is a triple $(M, +, \cdot, 0)$, where M is a set, $+$ and $\cdot$ are operations:

$$+ : M \times M \mapsto M \qquad\qquad \cdot : S \times M \mapsto M$$

$+$ is associative and commutative and $\cdot$ satisfies the usual associative and distributive properties of the product by a scalar. We will also require that

$$\forall v \in M, 1 \cdot v = v \qquad\qquad \forall v \in M, 0 \cdot v = 0$$

An S-semimodule is **zerosumfree** if $x + y = 0$ implies $x = y = 0$.

Some of the usual properties hold: $\forall a \in S, a \cdot 0 = 0$ and $\forall a \in S^*, \forall v \in M, a \cdot v = 0 \Rightarrow v = 0$. Most notions of linear algebra can be defined as usual, like **submodules**, **linear combinations**, the submodule **spanned** by a set A ($\mathrm{Span}(A)$), **linear maps**.

Let S be a semifield and M be an S-module. The **projective equivalence** relation on M is defined as:

$$x \sim y \Leftrightarrow \exists \lambda \in S^* : x = \lambda \cdot y$$

This is an equivalence relation. The **projective space** associated with M may be defined as the quotient by this relation:

$$\mathbb{P}(M) = (M \setminus \{0\})/\sim$$

The quotient map will be denoted by $\pi : M \setminus \{0\} \mapsto \mathbb{P}(M)$. The image by π of a submodule is a **projective subspace**.

If $f : M \mapsto N$ is a linear map, we note that $v \sim w \Rightarrow f(v) \sim f(w)$. The linear map induces a map between the associated projective spaces provided that the following condition holds:

$$\{v \in M \mid f(v) = 0\} \subset \{0\}$$

We will denote the induced map as $\overline{f} : \mathbb{P}(M) \mapsto \mathbb{P}(N)$. Maps of this kind will be called **projective maps**. The condition does not imply in general that the map is injective. Actually a projective map $\overline{f} : \mathbb{P}(M) \mapsto \mathbb{P}(M)$ may be not injective nor surjective in general.

The minimal number of elements required to span a semimodule is not a good indicator of its geometric dimension.

Definition 2.2. An S-semimodule M has dimension less than or equal to n if for every linear combination

$$v = a_1 \cdot v_1 + \cdots + a_s \cdot v_s$$

with $v, v_i \in M$, $a_i \in S$, $s > n$, it is possible to find indexes $i_1, \ldots, i_n \in \{1, \ldots, s\}$ and scalars $b_1, \ldots, b_n \in S$ such that

$$v = b_1 \cdot v_{i_1} + \ldots b_n \cdot v_{i_n}$$

An S-semimodule M has dimension n (written $\dim_S(M) = n$) if it has dimension less than or equal to n, and it does not have dimension less than or equal to $n-1$. The dimension of the projective space $\mathbb{P}(M)$ is defined as $\dim_S(\mathbb{P}(M)) = \dim_S(M) - 1$.

3. Examples

3.1. Free semimodules. The simplest example of S-semimodule is the **free** S-semimodule of **rank** n, i.e. the set S^n where the semigroup operation is the component-wise sum, and the product by a scalar is applied to every component. Note that if S is zerosumfree, then the semimodules S^n are zerosumfree too. S^n is spanned by n elements and it has dimension n.

Free semimodules have the usual universal property: let M be a S-semimodule, and $v_1, \ldots, v_n \in M$. Then there is a linear map:

$$S^n \ni c \mapsto c^1 \cdot v_1 + \cdots + c^n \cdot v_n \in \mathrm{Span}_S(v_1, \ldots, v_n)$$

This map sends e_i in v_i and its image is $\mathrm{Span}_S(v_1, \ldots, v_n)$.

Finitely generated semimodules are the semimodules admitting a finite set of generators. They are always finite dimensional, but the dimension is not always equal to the cardinality of a minimal set of generators. By the universal property, every finitely generated S-semimodule is the image of a free S-semimodule.

If S is zerosumfree, other examples are the following submodules of S^n:

$$FS^n = \mathrm{Span}_S((S^*)^n) = (S^*)^n \cup \{0\} \subset S^n$$

The projective space associated with S^n is $\mathbb{P}(S^n) = S\mathbb{P}^{n-1}$, and the projective space associated with FS^n is $\mathbb{P}(FS^n) = FS\mathbb{P}^{n-1}$.

$S\mathbb{P}^1 = \mathbb{P}(S^2)$ can be identified with $S \cup \{+\infty\}$ via the map:

$$S\mathbb{P}^1 \ni [x^1 : x^2] \mapsto x^1/x^2 \in S \cup \{+\infty\}$$

We give a name to three points: $0 = [0:1], 1 = [1:1], +\infty = [1:0] = +\infty$.

When $S = \mathbb{R}_{\geq 0}$ or $S = \mathbb{T}_{\mathbb{R}}$, $S\mathbb{P}^{n-1}$ may be described as an $(n-1)$-simplex, whose set of vertices is $\{\pi(e_1), \dots, \pi(e_n)\}$ (e_i being the elements of the canonical basis of S^n). Given a set of vertices A, the face with vertices in A is the projective subspace $\pi(\mathrm{Span}_S(A))$. $FS\mathbb{P}^{n-1}$ is naturally identified with the interior of the simplex $S\mathbb{P}^{n-1}$.

Let $f : S^n \mapsto S^m$ be a linear map. Then we can associate with f an m-by-n matrix with coefficients in S, as in standard linear algebra. While these matrices preserves all the usual formal properties, their geometric properties are very different. For example, is S is zerosumfree, there are very few bijective linear maps $S^n \mapsto S^n$. As we are mostly concerned with actions of groups over semimodules, this means that in the zerosumfree case free semimodules are not what we are searching for.

Let $\mathbb{F}$ be a field with a valuation, and let $\tau : \mathbb{F} \mapsto \mathbb{T}_\Lambda$ be its tropicalization map. We can extend this tropicalization map component-wise:

$$\tau : \mathbb{F}^n \mapsto \mathbb{T}^n_\Lambda$$

$$\tau : (\mathbb{F}_{\geq 0})^n \mapsto \mathbb{T}^n_\Lambda$$

Let $f : \mathbb{F}^n \mapsto \mathbb{F}^m$ be a linear map, expressed by a matrix $[f] = (a^i_j)$. Its tropicalization is the map $f^\tau : \mathbb{T}^n \mapsto \mathbb{T}^m$ defined by the matrix $[f^\tau] = (\alpha^i_j) = (\tau(a^i_j))$. Let $A \in GL_n(\mathbb{F})$ be an invertible matrix. Its tropicalization $\alpha = A^\tau : \mathbb{T}^n \mapsto \mathbb{T}^n$ (i.e. $\alpha = (\alpha^i_j) = (\tau(a^i_j))$) is, in general, not invertible. Anyway it induces a projective map $\mathbb{TP}^{n-1} \mapsto \mathbb{TP}^{n-1}$.

Now let $B = A^{-1}$, the inverse of A. We will write $\beta = B^\tau$. We would like to see β as an inverse of α, but this is impossible, as α is not always invertible. We will call **inversion domain** the set

$$D_{\alpha\beta} = \{x \in \mathbb{T}^n \mid \alpha(\beta(x)) = x\}$$

PROPOSITION 3.1. *The inversion domains have this name because of the following property: $D_{\beta\alpha} = \beta(D_{\alpha\beta})$, $D_{\alpha\beta} = \alpha(D_\beta\alpha)$ and $\beta_{|D_{\alpha\beta}} : D_{\alpha\beta} \mapsto D_{\beta\alpha}$ is bijective with inverse $\alpha_{|D_{\beta\alpha}} : D_{\beta\alpha} \mapsto D_{\alpha\beta}$.*

The set $D_{\alpha\beta}$ is a tropical submodule, and we can write explicit equations for it:

$$D_{\alpha\beta} = \{x \in \mathbb{T}^n \mid \forall h, k, x^h - x^k \geq (\alpha \odot \beta)^h_k\}$$

As a consequence, if $A \in GL_n(\mathcal{O})$, then $D_{\alpha\beta} \neq \emptyset$. Note that the matrices α and β are not inverse of each other, but, in the hypothesis $D_{\alpha\beta} \neq \emptyset$, then $\forall i, (\alpha \odot \beta)^i_i = 1_{\mathbb{T}}$.

PROOF. See [**1**]. □

3.2. Convex sets. The vector space $\mathbb{R}^n$ is an $\mathbb{R}_{\geq 0}$-semimodule, whose associated projective space over $\mathbb{R}_{\geq 0}$ can be identified with the sphere $\mathbb{S}^{n-1}$. If $\mathbb{F}$ is an ordered field, we denote by $\mathbb{FS}^{n-1}$ the projective space associated with $\mathbb{F}^n$ over $\mathbb{F}_{\geq 0}$. We denote the projections on the projective spaces by $\pi_{\mathbb{F}} : \mathbb{F}^n \setminus \{0\} \mapsto \mathbb{FP}^{n-1}$ and $\pi_{\mathbb{F}_{\geq 0}} : \mathbb{F}^n \setminus \{0\} \mapsto \mathbb{FS}^{n-1}$. There is also a natural 2-to-1 covering map $p : \mathbb{FS}^{n-1} \mapsto \mathbb{FP}^{n-1}$.

As $\mathbb{F}$ is ordered we can define the notion of convexity in $\mathbb{F}^n$: the **segment** with extremes $x, y \in \mathbb{F}^n$ is:

$$\sigma_{x,y} = \{\lambda x + (1-\lambda)y | \lambda \in \mathbb{F}, 0 \leq \lambda \leq 1\}$$

As usual a subset $\Omega \subset \mathbb{F}^n$ is **convex** if for all $x, y \in \Omega$, $\sigma_{x,y} \subset \Omega$.

A subset $C \subset \mathbb{F}^n$ is a **cone** if for every $x \in C$ and for every $\lambda \in \mathbb{F}_{>0}$, $\lambda x \in C$. The $\mathbb{F}_{\geq 0}$-submodules of $\mathbb{F}^n$ are precisely the convex cones containing 0. If $C \subset \mathbb{F}^n \setminus \{0\}$ is a convex cone, then $C \cup \{0\}$ is a zerosumfree $\mathbb{F}_{\geq 0}$-semimodule. An example is the **Minkowski cone**:

$$M = \{x \in \mathbb{R}^n \mid x_n^2 > x_1^2 + \cdots + x_{n-1}^2\}$$

Convex subsets of $\mathbb{F}\mathbb{P}^n$ are usually defined in the following way: an **affine space** in $\mathbb{F}\mathbb{P}^n$ is the complement of a projective hyperplane. A set $\Omega \subset \mathbb{F}\mathbb{P}^n$ is **convex** if it is contained in some affine space and its intersection with every projective line is connected. A convex set is **properly convex** if its closure $\overline{\Omega}$ is contained in an affine space. A properly convex set $\Omega \subset \mathbb{F}\mathbb{P}^n$ is **strictly convex** if its boundary $\partial\Omega$ does not contain any segment.

An equivalent definition is the following: **convex** subsets of the sphere $\mathbb{F}\mathbb{S}^{n-1}$ or $\mathbb{F}\mathbb{P}^{n-1}$ are the projective images of convex cones of $\mathbb{F}^n$ not containing 0. In other words, convex subsets of the sphere $\mathbb{F}\mathbb{S}^{n-1}$ are the projective spaces $\mathbb{P}(C)$, where C is a zerosumfree $\mathbb{F}_{\geq 0}$-submodule of $\mathbb{F}^n$. Also the convex subsets of $\mathbb{F}\mathbb{P}^{n-1}$ can be identified with projective spaces over $\mathbb{F}_{\geq 0}$, as the 2-to-1 map $p : \mathbb{F}\mathbb{S}^{n-1} \mapsto \mathbb{F}\mathbb{P}^{n-1}$ is always injective if restricted to a convex subset.

If C is a zerosumfree $\mathbb{F}_{\geq 0}$-submodule of $\mathbb{F}^n$, and $\Omega = \mathbb{P}(C)$ is a convex subset of $\mathbb{F}\mathbb{P}^n$, the group of projective automorphisms of $\mathbb{P}(C)$ over $\mathbb{F}_{\geq 0}$ is the group of projectivities of $\mathbb{F}\mathbb{P}^n$ preserving Ω. Such groups can be large Lie groups that act on Ω in a very interesting way. For example, if $C = M$, the Minkowski cone above, the corresponding projective space is $\mathbb{P}(M) = \mathbb{H}^n$, the Klein model of the hyperbolic space, and the group of projective automorphisms of $\mathbb{H}^n$ is $PO(1,n) \subset PGL_{n+1}(\mathbb{R})$, the group of hyperbolic isometries.

We need to construct projective spaces over the tropical semifields with properties that are similar to the properties of projective convex sets. Namely we need some projective spaces over the tropical semifields whose group of invertible projective map is large enough, such that there exists interesting actions of groups on the projective space. No subspace of $\mathbb{T}_\Lambda\mathbb{P}^{n-1}$ has this property, as subspaces of $\mathbb{T}_\Lambda\mathbb{P}^{n-1}$ are very rigid, with few invertible linear maps. For this reason we need to construct other projective spaces over $\mathbb{T}_\Lambda$, and to do this we will put a structure of tropical projective space on the Bruhat-Tits buildings.

If $C \subset \mathbb{F}^n$ is a finitely generated zerosumfree $\mathbb{F}_{\geq 0}$-semimodule, the corresponding projective space $\mathbb{P}(C)$ is a convex polytope. In this case, tropical analogues are known: if C is a finitely generated submodule of $\mathbb{T}_\mathbb{R}^n$, the projectivization $\mathbb{P}(C)$ is a subset of $\mathbb{T}_\mathbb{R}\mathbb{P}^{n-1}$, and these objects are called **tropical polytopes**. The relationships between convex polytopes over an ordered non-archimedean field and tropical polytopes are presented in [**8**]. For example, the image, under the tropicalization map, of a convex polytope over a non-archimedean field is a tropical polytope.

3.3. Bruhat-Tits buildings. Given a non-archimedean field $\mathbb{F}$ with a surjective real valuation, we are going to construct a family of tropical projective spaces we will call $P^{n-1}(\mathbb{F})$, or simply P^{n-1} when the field is well understood. This family arises as a generalization of the Bruhat-Tits buildings for SL_n to non-archimedean fields with surjective real valuation. In the usual case of a field with integral valuation, Bruhat and Tits constructed a polyhedral complex of dimension $n-1$ with an action of $SL_n(\mathbb{F})$. In the case $n=2$, Morgan and Shalen generalized this construction to a field with a general valuation, and they studied these objects using the theory of real trees. We want to extend this to general n, and we think that a good structure to study these objects is the structure of tropical projective spaces.

Let $V=\mathbb{F}^n$, an $\mathbb{F}$-vector space of dimension n and an infinitely generated $\mathcal{O}$-module. We consider the natural action $GL_n(\mathbb{F})\times V\mapsto V$.

Definition 3.2. An **$\mathcal{O}$-lattice** of V is an $\mathcal{O}$-finitely generated $\mathcal{O}$-submodule of V.

If L is a $\mathcal{O}$-finitely generated $\mathcal{O}$-submodule of V, then every minimal set of generators is $\mathbb{F}$-linearly independent, hence L is free. The rank of L is a number from 0 to n. A **maximal $\mathcal{O}$-lattice** is an $\mathcal{O}$-lattice of rank n.

We denote by $U^n(\mathbb{F})$ (or simply U^n) the set of all $\mathcal{O}$-lattices of $V=\mathbb{F}^n$, and by $FU^n(\mathbb{F})$ (or simply FU^n) the subset of all maximal $\mathcal{O}$-lattices and the $\mathcal{O}$-lattice $\{0\}$. U^n and FU^n can be turned in $\mathbb{T}$-semimodules by means of the following operations:

$\oplus : U^n\times U^n\mapsto U^n \qquad L\oplus M=\mathrm{Span}_{\mathcal{O}}(L\cup M)$

$\odot : \mathbb{T}\times U^n\mapsto U^n \qquad x\odot L=zL$, where $z\in\mathbb{F},\tau(z)=x$

We will denote by $\mathbb{P}(U^n(\mathbb{F}))=P^{n-1}(\mathbb{F})$ and $\mathbb{P}(FU^n(\mathbb{F}))=FP^{n-1}(\mathbb{F})$ the associated tropical projective spaces. We will simply write P^{n-1} and FP^{n-1} when the field $\mathbb{F}$ is understood.

As we said there is a natural action $GL_n(\mathbb{F})\times V\mapsto V$. Every element $A\in GL_n(\mathbb{F})$ sends $\mathcal{O}$-lattices in $\mathcal{O}$-lattices, hence we have an induced action $GL_n(\mathbb{F})\times U^n\mapsto U^n$. This action preserves the rank of a lattice, and in particular it sends FU^n in itself. Among the $\mathcal{O}$-lattices with the same rank this action is transitive, for example there exist an $A\in GL_n(\mathbb{F})$ sending every maximal $\mathcal{O}$-lattice of V in the standard lattice $\mathcal{O}^n\subset V$.

Hence the group $SL_n(\mathbb{F})$ acts naturally on U^n and FU^n by tropical linear maps and on P^{n-1} and FP^{n-1} by tropical projective maps.

Let $\mathcal{E}=(e_1,\dots,e_n)$ be a basis of V. We denote by $\varphi_{\mathcal{E}}:\mathbb{T}^n\mapsto U^n$ the map:

$$\varphi_{\mathcal{E}}(y)=\varphi_{\mathcal{E}}(y^1,\dots,y^n)=I_{y^1}e_1+\dots+I_{y^n}e_n=\mathrm{Span}_{\mathcal{O}}(t_{y^1}e_1,\dots t_{y^n}e_n)$$

The maps $\varphi_{\mathcal{E}}$ are injective and $\varphi_{\mathcal{E}}(F\mathbb{T}^n)\subset FU^n$. For every basis $\mathcal{E}$ we have a different map $\varphi_{\mathcal{E}}$. The union of the images of all these maps is the whole U^n, and the union of all the sets $\varphi(F\mathbb{T}^n)$ is equal to FU^n. We will call the maps $\varphi_{\mathcal{E}}$ **tropical charts** for U^n. Theorem 3.3 will justify this name. Given two points $x,y\in U^n$, there is a tropical chart containing both of them in its image.

Given two bases $\mathcal{E}=(e_1,\dots,e_n)$ and $\mathcal{F}=(f_1,\dots,f_n)$, we have two charts $\varphi_{\mathcal{E}},\varphi_{\mathcal{F}}$. We want to study the intersection of the images.

We put $I=\varphi_{\mathcal{E}}(\mathbb{T}^n)\cap\varphi_{\mathcal{F}}(\mathbb{T}^n)$, $I_{\mathcal{E}}=\varphi_{\mathcal{E}}^{-1}(I)$, $I_{\mathcal{F}}=\varphi_{\mathcal{F}}^{-1}(I)$. We want to describe the sets $I_{\mathcal{F}},I_{\mathcal{E}}$ and the **transition function**: $\varphi_{\mathcal{F}\mathcal{E}}=\varphi_{\mathcal{F}}^{-1}\circ\varphi_{\mathcal{E}}:I_{\mathcal{E}}\mapsto I_{\mathcal{F}}$.

The transition matrices between $\mathcal{E}$ and $\mathcal{F}$ are denoted by $A=(a^i_j),B=(b^i_j)\in GL_n(\mathbb{F})$:

$$\forall j, e_j=\sum_i a^i_j f_i \qquad \forall j, f_j=\sum_i b^i_j e_i \qquad A=B^{-1}$$

We will write $\alpha=A^\tau$ and $\beta=B^\tau$, i.e. $\alpha=(\alpha^i_j)=(\tau(a^i_j))$, $\beta=(\beta^i_j)=(\tau(b^i_j))$.

THEOREM 3.3 ([Description of the tropical charts]). *We have that* $I_{\mathcal{F}} = D_{\alpha\beta}$ *and* $I_{\mathcal{E}} = D_{\beta\alpha}$, *the inversion domains described in proposition 3.1. Moreover* $\varphi_{\mathcal{F}\mathcal{E}} = \alpha_{|I_{\mathcal{E}}}$ *and* $\varphi_{\mathcal{E}\mathcal{F}} = \beta_{|I_{\mathcal{F}}}$, *the tropicalizations of the transition matrices.*

PROOF. See [**1**]. □

4. Natural distances on projective spaces

The Hilbert metric is a distance defined on every properly convex subset $\Omega \subset \mathbb{RP}^n$. This distance is based on cross-ratios: if $x, y \in \Omega$, the projective line through x and y intersects $\partial\Omega$ in two points a, b. The distance is then defined as $d(x,y) = \frac{1}{2}\log[a,x,y,b]$ (order chosen such that $\overline{ax} \cap \overline{yb} = \emptyset$). If Ω, Ω' are convex subsets of $\mathbb{RP}^n$ and if $f : \Omega \mapsto \Omega'$ is the restriction of a projective map, then $d(f(x), f(y)) \leq d(x,y)$. In particular every projective isomorphism $f : \Omega \mapsto \Omega'$ is an isometry. Moreover this distance has straight lines as geodesics.

We can give an analogous definition for projective spaces over $\mathbb{T}_\mathbb{R}$. If M is a $\mathbb{T}_\mathbb{R}$-module there is a canonical way for defining a map

$$d : \mathbb{P}(M) \times \mathbb{P}(M) \mapsto \mathbb{R}_{\geq 0} \cup \{+\infty\}$$

with the following properties:

(1) $d(x,x) = 0$.
(2) $d(x,y) = d(y,x)$.
(3) $d(x,y) \leq d(x,z) + d(z,y)$.
(4) If $f : \mathbb{P}(M) \mapsto \mathbb{P}(N)$ is a projective map, then $d(f(x), f(y)) \leq d(x,y)$, and if $S \subset M$ is such that $f_{|S}$ is injective, then $f_{|S}$ is an isometry.

These maps fail to be distances because they can take the value $+\infty$, and because in some projective spaces they are **degenerate**, i.e. there are distinct points with 0 distance. We can give necessary and sufficient conditions on the projective space for this function to be non-degenerate. For example in the spaces $\mathbb{T}_\mathbb{R}\mathbb{P}^n$, $F\mathbb{T}_\mathbb{R}\mathbb{P}^n$, P^n and FP^n the distance is non degenerate. Moreover in $F\mathbb{T}_\mathbb{R}\mathbb{P}^n$ and FP^n it never takes the value $+\infty$, hence in these last two examples d is a distance in the ordinary sense.

This distance can be defined searching for a tropical analogue of the cross ratio. In $\mathbb{RP}^1$ the cross ratio can be defined by the identity $[0, 1, z, \infty] = z$ and the condition of being a projective invariant. Or equivalently if A is the (unique) projective map satisfying $A(0) = a, A(1) = b, A(\infty) = d$, then $[a,b,c,d] = A^{-1}(c)$. In this form the definition can be transposed to the tropical case.

Let $a = [a^1 : a^2], b = [b^1 : b^2], c = [c^1 : c^2], d = [d^1 : d^2] \in \mathbb{T}_\mathbb{R}\mathbb{P}^1 = \mathbb{P}(\mathbb{T}_\mathbb{R}^2)$ be points such that $a^1 - a^2 < b^1 - b^2 < c^1 - c^2 < d^1 - d^2$. There is a unique tropical projective map A satisfying $A(0_\mathbb{T}) = a, A(1_\mathbb{T}) = b, A(\infty_\mathbb{T}) = d$.

PROPOSITION 4.1. *The unique point* $x \in \mathbb{T}_\mathbb{R}\mathbb{P}^1$ *such that* $A(x) = c$ *is* $[(c^1 - c^2) - (b^1 - b^2) : 1_\mathbb{T}]$.

PROOF. See [**1**]. □

We can define the value $(c^1 - c^2) - (b^1 - b^2) \in \mathbb{R}$ as the **cross-ratio** of $[a,b,c,d]$. This value depends only on the central points b, c, and it is invariant by every tropical projective map $B : \mathbb{T}_\mathbb{R}\mathbb{P}^1 \mapsto \mathbb{T}_\mathbb{R}\mathbb{P}^1$ that is injective on the interval $[b,c]$. Consider a tropical projective map $B : \mathbb{T}_\mathbb{R}\mathbb{P}^1 \mapsto \mathbb{T}_\mathbb{R}\mathbb{P}^1$ such that $B(0_\mathbb{T}) = b$ and $B(\infty_\mathbb{T}) = c$. The inverse images $B^{-1}(b)$ and $B^{-1}(c)$ are, respectively, an initial segment and a final segment of $\mathbb{T}_\mathbb{R}\mathbb{P}^1$ with reference to the order $\preceq$ of $\mathbb{T}_\mathbb{R}\mathbb{P}^1$. This segments have an extremal point, b_0 and c_0 respectively. The restriction $B_{|[b_0,c_0]}$: $[b_0, c_0] \mapsto [b, c]$ is a projective isomorphism, hence $(c^1 - c^2) - (b^1 - b^2) = (c_0^1 - c_0^2) - (b_0^1 - b_0^2)$.

When we define the Hilbert metric we don't need to take the logarithms, as coordinates in tropical geometry already are in logarithmic scale. Hence the Hilbert metric on $\mathbb{T}_\mathbb{R}\mathbb{P}^1$ is simply the Euclidean metric:

$$d(x, y) = |(x^1 - x^2) - (y^1 - y^2)|$$

This definition can be extended to every tropical projective space $\mathbb{P}(M)$. If $a, b \in \mathbb{P}(M)$, we can choose two lifts $\bar{a}, \bar{b} \in M$. Then there is a unique linear map $\bar{f} : \mathbb{T}_\mathbb{R}^2 \mapsto M$ such that $f(e_1) = \bar{b}, f(e_2) = \bar{a}$. The induced projective map $f : \mathbb{T}_\mathbb{R}\mathbb{P}^1 \mapsto \mathbb{P}(M)$ sends $0_\mathbb{T}$ in a and $\infty_\mathbb{T}$ in b. As before, the sets $f^{-1}(a)$ and $f^{-1}(b)$ are closed segments, with extremal points a_0, b_0. We can define the distance as $d(a, b) = d(a_0, b_0)$. It is easy to verify that this definition does not depend on the choice of the lifts $\bar{a}, \bar{b}$.

Proposition 4.2. *The distance d satisfies the properties 1,2,3,4 stated above.*

Proof. See [**1**]. □

For the projective spaces associated with the free modules we can calculate explicitly this distance. Let $x, y \in \mathbb{T}_\mathbb{R}\mathbb{P}^{n-1}$. Then, for all lifts $\bar{x}, \bar{y} \in \mathbb{T}_\mathbb{R}^n$:

$$d(x, y) = \left(\bigoplus_{i=1}^{n} \bar{x}^i \oslash \bar{y}^i\right) \odot \left(\bigoplus_{i=1}^{n} \bar{y}^i \oslash \bar{x}^i\right) = \max_{i=1}^{n}(\bar{x}^i - \bar{y}^i) + \max_{i=1}^{n}(\bar{y}^i - \bar{x}^i)$$

This is a well known distance, the Hilbert metric on the simplex in logarithmic coordinates.

Now we show a pathological example. Consider the following equivalence relation on $\mathbb{T}_\mathbb{R}^2$:

$$(x^1, x^2) \sim (y^1, y^2) \Leftrightarrow \begin{cases} x^1 < x^2, y^1 < y^2 \text{ and } x^2 = y^2 \\ \textit{or} \\ x^1 \geq x^2, y^1 \geq y^2 \text{ and } x^1 = y^1 \end{cases}$$

The quotient for this relation will be denoted by B. If $a \sim a'$ and $b \sim b'$, then $a \oplus b = a' \oplus b'$ and $\lambda \odot a = \lambda \odot a'$. Hence the operations $\oplus$, $\odot$ induces operations on B, turning it in a finitely generated $\mathbb{T}_\mathbb{R}$-semimodule. We will denote the equivalence classes in the following way: if (x^1, x^2) satisfies $x^1 < x^2$ we will denote its class as $[(\cdot, x^2)]$, if $x^1 \geq x^2$ we will denote its class as $[(x^1, \cdot)]$. The distance d is degenerate on this projective space, as $d([(x^1, \cdot)], [(\cdot, x^1)]) = 0$. This is, in some sense, the only example with a degenerate distance.

If we put on the quotient a topology making the projection continuous, then the point $[(x^1, \cdot)]$ is not closed, as its closure must contain the point $[(\cdot, x^1)]$. We define a $\mathbb{T}_\mathbb{R}$-semimodule to be **separated** if it does not contain any submodule isomorphic to B.

Proposition 4.3. *The distance d is non degenerate if and only if the projective space is separated.*

Proof. See [**1**]. □

Examples of separated $\mathbb{T}_{\mathbb{R}}$-semimodules are the free semimodules (since there exists no submodule in $\mathbb{T}_{\mathbb{R}}^n$ whose associated projective space has exactly two points) and the semimodules U^n (since every two points in U^n are in the image of the same tropical chart, hence in a submodule isomorphic to $\mathbb{T}_{\mathbb{R}}^n$).

The metric we have defined for separated tropical projective spaces can achieve the value $+\infty$. Given a $\mathbb{T}_{\mathbb{R}}$-semimodule M we can define the following equivalence relation on $M \setminus \{0\}$:

$$x \sim y \Leftrightarrow d(\pi(x), \pi(y)) < +\infty$$

The union of $\{0\}$ with one of these equivalence classes is again a $\mathbb{T}_{\mathbb{R}}$-semimodule, and their projective quotients are tropical projective spaces with an ordinary (i.e. finite) metric.

For example in the free $\mathbb{T}_{\mathbb{R}}$-semimodules $\mathbb{T}_{\mathbb{R}}^n$ the equivalence class of the point $(1_{\mathbb{T}}, \dots, 1_{\mathbb{T}})$ is the set $F\mathbb{T}_{\mathbb{R}}^n$, and its associated projective space is $F\mathbb{T}_{\mathbb{R}}\mathbb{P}^{n-1}$, a tropical projective space in which the metric is finite.

For the $\mathbb{T}_{\mathbb{R}}$-semimodule U^n an equivalence class is FU^n, and its associated projective space is FP^{n-1}, a tropical projective space in which the metric is finite.

We can calculate more explicitly the metric for FP^{n-1}. Let $x, y \in FP^{n-1}$ and let $\bar{x}, \bar{y} \in U^n$ be their lifts. Choose a tropical chart $\varphi_{\mathcal{E}}$ containing x, y. Up to translation it is possible to have that $\mathcal{E} = (e_1, \dots, e_n)$ is a basis of $\bar{x}$ and $a_1 e_1, \dots a_n e_n$ is a basis of $\bar{y}$. In the tropical chart $\varphi_{\mathcal{E}}$, the point $\bar{x}$ has coordinates $(1_{\mathbb{T}}, \dots, 1_{\mathbb{T}})$, while the point $\bar{y}$ has coordinates $(\tau(a_1), \dots, \tau(a_n))$. Hence

$$d(x, y) = \max_i(\tau(a_i)) - \min_i(\tau(a_i))$$

Proposition 4.4. *For every separated $\mathbb{T}$-module M, its associated projective space $\mathbb{P}(M)$ is contractible with reference to the topology induced by the canonical metric.*

Proof. See [**1**]. □

5. Real and Tropical Projective Manifolds

5.1. Convex real projective manifolds. Let M be an n-manifold. A **coordinate chart** taking values in $\mathbb{RP}^n$ is a pair (U, ϕ), where $U \subset M$ is open, and $\phi : U \mapsto \mathbb{RP}^n$ is a diffeomorphism with its image, an open subset of $\mathbb{RP}^n$. If the domains of two coordinate charts (U, ϕ) and (V, ψ) intersect, the **transition map** between them is

$$\phi_{|U\cap V} \circ (\psi^{-1})_{|\psi(U\cap V)} : \psi(U \cap V) \mapsto \phi(U \cap V)$$

Note that $\psi(U \cap V), \phi(U \cap V) \subset \mathbb{RP}^n$. Two coordinate charts are **projectively compatible** if their domains don't intersect or if the transition map between them is a **locally projective map**. This means that for every connected component C of the intersection $U \cap V$ there exists a projective map $A \in PGL_{n+1}(\mathbb{R})$ such that

$$\left(\phi_{|U\cap V} \circ (\psi^{-1})_{|\psi(U\cap V)}\right)_{|C} = A_{|C}$$

A **real projective atlas** on M is a collection of charts that are pairwise projectively compatible and such that their domains cover M. A **real projective structure** on M is a maximal real projective atlas on M. A **real projective manifold** is a manifold together with a real projective structure. If M and N are projective manifolds, a diffeomorphism $f : M \mapsto N$ is a **projective isomorphism** if for each pair of charts $\phi : U \mapsto \mathbb{RP}^n$, $\psi : V \mapsto \mathbb{RP}^n$ (where $U \subset M$ and $V \subset N$) the map

$$\psi \circ f_{|f^{-1}(V)} \circ (\phi^{-1})_{|\phi(U \cap f^{-1}(V))} : \phi(U \cap f^{-1}(V)) \mapsto \mathbb{RP}^n$$

is a locally projective map.

For the general theory of geometric structures on manifolds, see [**10**]. Here we will need the notions of development map and holonomy. The development map is a global version of the local coordinate charts. Let $\pi : \widetilde{M} \mapsto M$ be the universal covering of M. If M has a real projective structure, a **development map** for the structure is a local diffeomorphism $D : \widetilde{M} \mapsto \mathbb{RP}^n$ such that every $x \in \widetilde{M}$ has an open neighborhood U such that $D_{|U}$ and $\pi_{|U}$ are injective and $D \circ (\pi_{|U})^{-1}$ is a coordinate chart for $\pi(U)$. We identify the fundamental group $\pi_1(M)$ with the group of deck transformations of the covering space. Then there exists an homomorphism $h : \pi_1(M) \mapsto PGL_{n+1}(\mathbb{R})$ such that for every $\gamma \in \pi_1(M)$ we have $h(\gamma) \circ D = D \circ \gamma$. The pair (D, h) is called a **development pair** for the structure, and the homomorphism h is called a **holonomy representation**. The development pair is unique in the following sense: if (D', h') is another such pair, there exist $g \in PGL_{n+1}(\mathbb{R})$ such that $D' = g \circ D$ and for all $\gamma \in \pi_1(M)$, $h'(\gamma) = gh(\gamma)g^{-1}$. A development pair determines the real projective structure on M. A general theorem guarantees the existence of a developing pair for every real projective structure, see [**10**].

The most important examples of real projective manifolds are given by hyperbolic manifolds. According to the Klein model of hyperbolic space, the hyperbolic space is identified with an ellipsoid $\mathbb{H}^n \subset \mathbb{RP}^n$, and the group of hyperbolic isometries is identified with the group of projective transformations of the ellipsoid, $O^+(1, n) \subset PGL_{n+1}(\mathbb{R})$. Hence, every hyperbolic manifold has a canonical real projective structure.

If the hyperbolic manifold is complete, it is the quotient of a discrete subgroup of $O^+(1, n)$ acting properly and freely on $\mathbb{H}^n$. Convex real projective manifolds are a generalization of this construction, and they share many properties with hyperbolic manifolds.

A **convex real projective manifold** is a projective manifold M isomorphic to Ω/Γ, where $\Omega \subset \mathbb{RP}^n$ is an open properly convex domain and $\Gamma \subset PGL_{n+1}(\mathbb{R})$ is a discrete group acting properly and freely on Ω. In other words, a projective structure is convex if an only if the developing map is injective, with image a properly convex open subset of $\mathbb{RP}^n$. Hence the development map identifies $\widetilde{M}$ with Ω, and the holonomy representation identifies $\pi_1(M)$ with Γ. A **strictly convex projective manifold** is a convex projective manifold Ω/Γ, where Ω is strictly convex.

THEOREM 5.1. *Let $\Gamma \subset PGL_{n+1}(\mathbb{R})$ be a discrete subgroup acting on a properly convex open set $\Omega \subset \mathbb{RP}^n$. Then*

(1) *The action of Γ on Ω is proper.*
(2) *The action of Γ on Ω is free (or, equivalently, the quotient map $\Omega \mapsto \Omega/\Gamma$ is a covering) if and only if Γ is torsion-free.*
(3) *If the quotient Ω/Γ is compact, then Ω is strictly convex if and only if Γ is Gromov hyperbolic.*

PROOF. See [**4**] and [**5**]. □

5.2. Tropical Projective Manifolds. In the following we will work with a compact orientable n-manifold M (without boundary) such that its universal covering is $\mathbb{R}^n$ and its fundamental group is Gromov hyperbolic. Note that the conditions implies that the fundamental group is also torsion-free. The most important examples of such manifolds are given by closed orientable hyperbolic manifolds, as they are quotients of $\mathbb{H}^n$, that is a strictly convex set. The hypothesis on M implies that every convex projective structure on M is strictly convex.

A convex real projective structure on M is determined by its developing pair (D, h). Note that the development map is h-**equivariant** with respect to the natural action of $\pi_1(M)$ on $\widetilde{M}$ i.e. $\forall \gamma \in \pi_1(M), \forall x \in \widetilde{M}, D(\gamma(x)) = h(\gamma)(D(x))$ (for short $h(\gamma) \circ D = D \circ \gamma$).

Vice versa, if D is a diffeomorphism from $\widetilde{M}$ to an open projective subspace $\Omega \subset \mathbb{RP}^n$ over $\mathbb{R}_{\geq 0}$ (a convex subset), and h is a representation of $\pi_1(M)$ in the group of projective automorphisms of Ω, and D is h-equivariant, then there exists a convex projective structure on M whose development pair is (D, h).

This definition can be extended to the tropical world:

Definition 5.2. A **tropical projective structure** on M is given by a pair (D, h), where $D : \widetilde{M} \mapsto P$ is a continuous map from $\widetilde{M}$ to a projective space over $\mathbb{T}_{\mathbb{R}}$ of dimension n (continuous with reference to the topology induced by the natural distance), and h is a representation of $\pi_1(M)$ on the group of projective isomorphisms of P, and D is h-equivariant.

In this definition we don't require properties of regularity for D, as the idea is that tropical projective structures represent degenerate real projective structures, so we need to admit singularities. Actually to construct a tropical projective structure, only the representation is needed, the equivariant map comes automatically.

Theorem 5.3. *Let M be an n-manifold whose universal covering is $\mathbb{R}^n$. Let h be a representation of $\pi_1(M)$ in the group of projective isomorphisms of a projective space P over $\mathbb{T}_{\mathbb{R}}$. Then there exists a map $f : \widetilde{M} \mapsto P$ that is h-equivariant.*

Proof. See [**1**]. $\square$

5.3. Length spectra. Let $SL^{\pm}_{n+1}(\mathbb{R}) \subset GL_{n+1}(\mathbb{R})$ be the subgroup of matrices with determinant ± 1. Then $PGL_{n+1} = SL^{\pm}_{n+1}(\mathbb{R})/\{\pm \mathrm{Id}\}$.

If $\gamma \in PGL_{n+1}(\mathbb{R})$, let $\overline{\gamma} \in SL^{\pm}_{n+1}(\mathbb{R})$ be a lift. Let $\lambda_1(\gamma), \ldots, \lambda_{n+1}(\gamma)$ be its complex eigenvalues, ordered such that $|\lambda_1(\gamma)| \geq |\lambda_2(\gamma)| \geq \cdots \geq |\lambda_{n+1}(\gamma)|$. The element γ is said to be **proximal** if $|\lambda_1(\gamma)| > |\lambda_2(\gamma)|$. In this case $\lambda_1(\gamma)$ is real, and its eigenvector corresponds to the unique attracting fixed point $x_\gamma \in \mathbb{RP}^n$ of γ.

Proposition 5.4. *Let $\Gamma \subset PGL_{n+1}(\mathbb{R})$ be a torsion-free group dividing a strictly convex set Ω. Then every element $\gamma \in \Gamma$ is proximal. In particular γ^{-1} is also proximal, hence the eigenvector $\lambda_{n+1}(\gamma)$ is real. Moreover, if $\overline{\gamma} \in SL^{\pm}_{n+1}(\mathbb{R})$ is a lift of γ, then $\lambda_1(\overline{\gamma})$ and $\lambda_{n+1}(\overline{\gamma})$ have the same sign.*

Proof. See [**5**]. $\square$

The point $y_\gamma = x_{\gamma^{-1}}$ is the unique repelling fixed point of γ. The points x_γ, y_γ are in $\partial\Omega$, and the segment (x_g, y_g) is the unique invariant geodesic of γ in Ω. The image of (x_γ, y_γ) in Ω/Γ is the unique geodesic in the free-homotopy class of γ. Moreover, Ω/Γ does not contain any closed homotopically trivial geodesic.

Corollary 5.5. *The set $\pi^{-1}(\Omega) \subset \mathbb{R}^{n+1}$ is the union of two convex cones. The group Γ can be lifted to a subgroup $\overline{\Gamma}$ of $SL^{\pm}_{n+1}(\mathbb{R})$ preserving each of the convex cones. After this lift, if $\gamma \in \Gamma$, then λ_1 and λ_{n+1} are real and positive.*

Let $\Omega \subset \mathbb{RP}^n$ be a properly convex set, and let $M = \Omega/\Gamma$ be a strictly convex projective manifold. Every $\gamma \in \Gamma$ acts on Ω as an isometry for the Hilbert distance. The **translation length** of γ is defined as

$$\ell_\gamma = \inf_{x\in\Omega} d(x, \gamma(x))$$

Geometrically, the element γ acts on the invariant geodesic (x_γ, y_γ) as a translation of length ℓ_γ. The translation length ℓ_γ can be computed from the eigenvalues λ_1 and λ_{n+1} by

$$\ell_\gamma = \log_e\left(\frac{\lambda_1}{\lambda_n}\right)$$

The function $\ell : \Gamma \mapsto \mathbb{R}_{>0}$ is called the **marked length spectrum** of M.

The marked length spectrum can be defined also for tropical projective structures constructed using the buildings, and it can be computed from eigenvalues of matrices in a similar way. Let $\mathbb{F}$ be a non-archimedean field with surjective real valuation, let Γ be a group and $\rho : \Gamma \mapsto GL_{n+1}(\mathbb{F})$ be a representation.

The group $GL_{n+1}(\mathbb{F})$ acts by linear maps on the tropical modules $U^{n+1}(\mathbb{F})$ and $FU^{n+1}(\mathbb{F})$, and by tropical projective maps on the tropical projective spaces $P^n(\mathbb{F})$ and $FP^n(\mathbb{F})$. The representation ρ defines an action of Γ on $FP^n(\mathbb{F})$.

Every matrix $A \in GL_{n+1}(\mathbb{F})$ acts on $FP^n(\mathbb{F})$ as an isometry for the natural distance, and we can define the **translation length** of A by:

$$l(A) = \inf_{x\in FP^n(\mathbb{F})} d(x, Ax)$$

The case $n = 1$ has been studied in [**12**]. If $A \in SL_2(\mathbb{F})$, we have $l(A) = 2\max(0, \tau(\mathrm{tr}(A)))$ (see [**12**, prop. II.3.15]). In the following we give an extension of this result for generic n.

Let $\mathbb{F}$ be a non-archimedean real closed field of finite rank extending $\mathbb{R}$, with a surjective real valuation $\overline{v} : \mathbb{F}^* \mapsto \mathbb{R}$ such that the valuation ring is convex. The field $\mathbb{K} = \mathbb{F}[i]$ is an algebraically closed field extending $\mathbb{C}$, with an extended valuation $\overline{v} : \mathbb{K}^* \mapsto \mathbb{R}$. We will use the notation $\tau = -\overline{v}$. We will also use the complex norm $|\cdot| : \mathbb{K} \mapsto \mathbb{F}_{\geq 0}$ defined by $|a + bi| = \sqrt{a^2 + b^2}$.

If $A \in GL_{n+1}(\mathbb{K})$, we denote by $\lambda_1, \ldots, \lambda_{n+1}$ its eigenvalues, ordered such that $|\lambda_i| \geq |\lambda_{i+1}|$.

Proposition 5.6. *Let $k = \mathbb{F}$ or $\mathbb{K}$. Then $A \in GL_{n+1}(k)$ acts on $FP^n(k)$. Then the* inf *in the definition of $l(A)$ is a minimum, and it is equal to*

$$l(A) = \tau\left(\left|\frac{\lambda_1}{\lambda_{n+1}}\right|\right)$$

Proof. See [**1**]. □

If (D, h) is a tropical projective structure on a manifold M, with $h : \pi_1(M) \mapsto SL_{n+1}(\mathbb{F})$, and $f : \widetilde{M} \mapsto FP^n(\mathbb{F})$ an h-equivariant map, we define the **marked length spectrum** of (D, h) as the function:

$$\ell : \Gamma\gamma \mapsto l(\rho(\gamma)) \in \mathbb{R}_{\geq 0}$$

6. Varieties of representations and of characters

Let Γ be a group and $\mathbb{K}$ a field of characteristic 0. A **representation** of Γ is a group homomorphism $\rho : \Gamma \mapsto GL_n(\mathbb{K})$.

A representation ρ is **absolutely irreducible** if it is irreducible with reference to the algebraic closure of $\mathbb{K}$, else it is **absolutely reducible**.

The **character** of a representation ρ is the function

$$\chi_\rho : \Gamma \ni \gamma \mapsto \operatorname{tr}(\rho(\gamma)) \in \mathbb{K}$$

By the conjugation-invariance of the trace, two conjugated representations have the same character. A sort of converse holds: let ρ, ρ' be two representations, and suppose that ρ is absolutely irreducible. Then they are conjugated if and only if they have the same character. See also [**13**, thm. 6.12] for a more general statement.

In the following Γ is assumed to be a finitely generated group. We will work with the group $SL_n^{\pm}$ (the group of matrices whose determinant is ± 1) but everything we say also holds for SL_n. When we write $SL_n^{\pm}$ we mean it as a scheme, an affine algebraic group, and we will denote by $SL_n^{\pm}(\mathbb{K})$ the set of $\mathbb{K}$-points of $SL_n^{\pm}$. There exists an affine $\mathbb{Q}$-algebraic scheme $\operatorname{Hom}(\Gamma, SL_n^{\pm})$ such that for every field $\mathbb{K}$, the set of $\mathbb{K}$-points $\operatorname{Hom}(\Gamma, SL_n^{\pm}(\mathbb{K}))$ is in natural bijection with the set of all representations of Γ in $SL_n^{\pm}(\mathbb{K})$.

The $\mathbb{Q}$-algebraic group PGL_n acts on $SL_n^{\pm}$ by conjugation, and this action induces an action on $\operatorname{Hom}(\Gamma, SL_n^{\pm})$. Every $\gamma \in \Gamma$ defines a polynomial function

$$\tau_\gamma : \operatorname{Hom}(\Gamma, SL_n^{\pm}(\mathbb{K})) \ni \rho \mapsto \chi_\rho(\gamma) \in \mathbb{K}$$

these functions belongs to the ring of coordinates of $\operatorname{Hom}(\Gamma, SL_n^{\pm})$), and they will be called **trace functions**. The trace functions are invariant for the action of PGL_n

There exists a closed subscheme $\operatorname{Hom}(\Gamma, SL_n^{\pm})_{a.r.r.}$ of $\operatorname{Hom}(\Gamma, SL_n^{\pm})$ whose set of $\mathbb{K}$ points $\operatorname{Hom}(\Gamma, SL_n^{\pm}(\mathbb{K}))_{a.r.r.}$ is the subset of all absolutely reducible representations (see [**13**]). We define also the open subscheme $\operatorname{Hom}(\Gamma, SL_n^{\pm})_{a.i.r}$ as the complement of $\operatorname{Hom}(\Gamma, G)_{a.r.r.}$, the set of absolutely irreducible representations.

Consider the action by conjugation of $PGL_n(\mathbb{K})$ on $\operatorname{Hom}(\Gamma, SL_n^{\pm}(\mathbb{K}))$. We denote by A the ring of coordinates of $\operatorname{Hom}(\Gamma, SL_n^{\pm})$, and by A_0 the subring of invariant functions for the action of PGL_n. As PGL_n is reductive, by [**11**, Chap. 1, thm. 1.1], the ring A_0 is finitely generated as a $\mathbb{Q}$-algebra. Note that the trace functions τ_γ belong to A_0. There exists a finite set $C \subset \Gamma$ such that the functions $\{\tau_\gamma\}_{\gamma \in C}$ generate A_0 (see [**15**]). Consider the map

$$t : \operatorname{Hom}(\Gamma, SL_n^{\pm}(\mathbb{K})) \ni \rho \mapsto \tau_\gamma(\rho)_{\gamma \in C} \in \mathbb{K}^{\operatorname{Card}(C)}$$

We will denote by $\operatorname{Char}(\Gamma, SL_n^{\pm}(\mathbb{K}))$ the Zariski closure of the image of this map, an affine $\mathbb{Q}$-algebraic set whose ring of coordinates is isomorphic to A_0.

The map t is dual to the inclusion map $A_0 \mapsto A$, hence it is identified with the semi-geometric quotient $\operatorname{Hom}(\Gamma, SL_n^{\pm}) = \operatorname{Spec}(A) \mapsto \operatorname{Spec} A_0$ as in [**11**, Chap. 1, thm. 1.1]. As this semi-geometric quotient is surjective, the image of the map t above is the set $\operatorname{Char}(\Gamma, SL_n^{\pm}(\mathbb{K}))$. We will write $\operatorname{Char}(\Gamma, SL_n^{\pm}) = \operatorname{Spec}(A_0)$.

If $C' \subset \Gamma$ is another finite set of generators, the pair $(\operatorname{Char}(\Gamma, SL_n^{\pm}), t)$ defined by C' is isomorphic to the previous one, hence this construction does not depend on the choices.

The functions $\{\tau_\gamma\}_{\gamma\in C}$ determine the values of all the trace functions $\{\tau_\gamma\}_{\gamma\in\Gamma}$, hence, if ρ is a representation, the point $t(\rho)$ determines the character χ_ρ. Hence the points of $\mathrm{Char}(\Gamma, SL_n^\pm(\mathbb{K}))$ are in natural bijection with the characters of the representations in $\mathrm{Hom}(\Gamma, SL_n^\pm(\mathbb{K}))$, and for this reason the affine $\mathbb{Q}$-algebraic set $\mathrm{Char}(\Gamma, SL_n^\pm(\mathbb{K}))$ will be called the **variety of characters**.

Consider the invariant subset $\mathrm{Hom}(\Gamma, SL_n^\pm(\mathbb{K}))_{a.i.r}$ of absolutely irreducible representations. The image of this set through the map t is open, and will be denoted by $\mathrm{Char}(\Gamma, SL_n^\pm(\mathbb{K}))_{a.i.r.}$. This is the set of $\mathbb{K}$-points of an algebraic scheme $\mathrm{Char}(\Gamma, SL_n^\pm)_{a.i.r.}$.

Consider the restriction of t to $\mathrm{Hom}(\Gamma, SL_n^\pm)_{a.i.r.}$:

$$t_{a.i.r.} : \mathrm{Hom}(\Gamma, SL_n^\pm)_{a.i.r.} \mapsto \mathrm{Char}(\Gamma, SL_n^\pm)_{a.i.r.}$$

This is a geometric quotient (see [**11**] for the definition), hence the set of its $\mathbb{K}$-points $\mathrm{Char}(\Gamma, SL_n^\pm(\mathbb{K}))_{a.i.r.}$ is in natural bijection with the set-theoretical quotient $\mathrm{Hom}(\Gamma, SL_n^\pm(\mathbb{K}))_{a.i.r.}/PGL_n(\mathbb{K})$. Actually $\mathrm{Hom}(\Gamma, SL_n^\pm)_{a.i.r.} \subset \mathrm{Hom}(\Gamma, SL_n^\pm)$ is precisely the subset of **properly stable points** for the action of PGL_n with respect to the canonical linearization of the trivial line bundle (see [**11**, Chap. 1, def. 1.8] and [**13**, rem. 6.6]).

We need a similar construction for a real closed field $\mathbb{F}$. The set of characters of representations $\rho : \Gamma \mapsto SL_n^\pm(\mathbb{F})$ is not an affine algebraic set in general. In this case we can prove that this set is a closed semi-algebraic set, and that the map $t : \mathrm{Hom}(\Gamma, SL_n^\pm(\mathbb{F})) \mapsto \mathrm{Char}(\Gamma, SL_n^\pm(\mathbb{F}))$ has properties similar to the properties it has in the algebraically closed case.

Let $\mathbb{K} = \mathbb{F}[i]$, the algebraic closure of $\mathbb{F}$. If $\mathrm{Hom}(\Gamma, SL_n^\pm(\mathbb{K})) \subset \mathbb{K}^m$ is an embedding defined over $\mathbb{Q}$, we have $\mathrm{Hom}(\Gamma, SL_n^\pm(\mathbb{F})) = \mathrm{Hom}(\Gamma, SL_n^\pm(\mathbb{K})) \cap \mathbb{F}^n$, and, in the same way, if $\mathrm{Char}(\Gamma, SL_n^\pm(\mathbb{K})) \subset \mathbb{K}^s$ is an embedding defined over $\mathbb{Q}$, we have $\mathrm{Char}(\Gamma, SL_n^\pm(\mathbb{F})) = \mathrm{Char}(\Gamma, SL_n^\pm(\mathbb{K})) \cap \mathbb{F}^s$.

The map $t : \mathrm{Hom}(\Gamma, SL_n^\pm(\mathbb{K})) \mapsto \mathrm{Char}(\Gamma, SL_n^\pm(\mathbb{K}))$ is defined over $\mathbb{Q}$, hence $t(\mathrm{Hom}(\Gamma, SL_n^\pm(\mathbb{F})) \subset \mathrm{Char}(\Gamma, SL_n^\pm(\mathbb{F}))$. Anyway $t(\mathrm{Hom}(\Gamma, SL_n^\pm(\mathbb{F})))$ is not in general the whole $\mathrm{Char}(\Gamma, SL_n^\pm(\mathbb{F}))$. For example an irreducible representation of Γ in $SU_2(\mathbb{C})$ has real character, but it is not conjugated to a representation in $SL_2^\pm(\mathbb{R})$ (see [**12**, prop. III.1.1] and the discussion for details). Hence the $\mathbb{F}$-algebraic set $\mathrm{Char}(\Gamma, SL_n^\pm(\mathbb{F}))$ is not in natural bijection with the set of characters of representations in $\mathrm{Hom}(\Gamma, SL_n^\pm(\mathbb{F}))$. We will denote by $\overline{\mathrm{Char}}(\Gamma, SL_n^\pm(\mathbb{F}))$ the image of $t_{|\,\mathrm{Hom}(\Gamma, SL_n^\pm(\mathbb{F}))}$, the actual set of characters of representations in $\mathrm{Hom}(\Gamma, SL_n^\pm(\mathbb{F}))$.

THEOREM 6.1. *Let $R \subset \mathrm{Hom}(\Gamma, SL_n^\pm(\mathbb{F})) \subset \mathbb{F}^m$ be a closed semi-algebraic set that is invariant for the action of $PGL_n(\mathbb{F})$. Then the image $t(R)$ under the semi-geometric quotient map t is a closed semi-algebraic subset of $\mathbb{F}^s$. In particular the set $\overline{\mathrm{Char}}(\Gamma, SL_n^\pm(\mathbb{F}))$ is a closed semi-algebraic set in natural bijection with the set of characters of representations in $\mathrm{Hom}(\Gamma, SL_n^\pm(\mathbb{F}))$, and the set $\overline{\mathrm{Char}}(\Gamma, SL_n^\pm(\mathbb{F}))_{a.i.r.}$ is in natural bijection with the set theoretical quotient $\mathrm{Hom}(\Gamma, SL_n^\pm(\mathbb{F}))_{a.i.r.}/PGL_n(\mathbb{F})$.*

PROOF. See [**3**]. □

7. Parameter spaces of projective structures

Let M be an n-manifold. A **marked** $\mathbb{RP}^n$**-structure** on M is a pair (N, ϕ), where N is an $\mathbb{RP}^n$-manifold and $\phi : M \mapsto N$ is a diffeomorphism. The diffeomorphism ϕ induces an $\mathbb{RP}^n$-structure on S. Two marked $\mathbb{RP}^n$-structures (N, ϕ), (N', ϕ') on M are **isotopic** if there is a projective isomorphism $h : N \mapsto N'$ such that ϕ' is isotopic to $h \circ \phi$.

We choose a base point $m_0 \in M$ and a universal covering space $\widetilde{M} \mapsto M$. A **based** $\mathbb{RP}^n$**-structure** on M is a triple (N, ϕ, D, h) where N is an $\mathbb{RP}^n$-manifold, $\phi : M \mapsto N$ is a diffeomorphism and (D, h) is a development pair for N. This developing pair induces, via the diffeomorphism ϕ, a developing pair (f, ρ) for the $\mathbb{RP}^n$-structure on M, such that $\rho : \pi_1(M, m_0) \mapsto PGL_n(\mathbb{R})$ is a representation, and $f : \widetilde{M} \mapsto \mathbb{RP}^n$ is a ρ-equivariant local diffeomorphism. Vice versa every such pair (f, ρ) determines a based $\mathbb{RP}^n$-structure on M.

We say that two based $\mathbb{RP}^n$-structures (f, ρ) and (f', ρ') are **isotopic** if $\rho = \rho'$ and there exists a diffeomorphism $h : (M, m_0) \mapsto (M, m_0)$, isotopic to the identity, such that $f' = f \circ \widetilde{h}$, where $\widetilde{h}$ is the lift of h to $\widetilde{M}$.

We consider the algebraic set $\mathrm{Hom}(\pi_1(M, m_0), PGL_n(\mathbb{R}))$ with the topology induced by the order topology of $\mathbb{R}$, and the set $C^\infty(\widetilde{M}, \mathbb{RP}^n)$ of smooth maps $\widetilde{M} \mapsto \mathbb{RP}^n$ with the C^∞ topology.

We define the deformation set of based $\mathbb{RP}^n$-structures:

$$\begin{gathered}\mathcal{D}'_{\mathbb{RP}^n}(M) = \{(f, \rho) \in C^\infty(\widetilde{M}, \mathbb{RP}^n) \times \mathrm{Hom}(\pi_1(M, m_0), PGL_n(\mathbb{R})) \mid \\ f \text{ is a } \rho\text{-equivariant local diffeomorphism}\}\end{gathered}$$

This set inherits the subspace topology. We denote by $\mathrm{Diff}(M, m_0)$ the group of all diffeomorphisms $M \mapsto M$ fixing m_0, and by $\mathrm{Diff}_0(M, m_0)$ the subgroup of all diffeomorphisms fixing m_0 and isotopic to the identity. The group $\mathrm{Diff}_0(M, m_0)$ acts properly and freely on $\mathcal{D}'_{\mathbb{RP}^n}(M)$. We denote by $\mathcal{D}_{\mathbb{RP}^n}(M)$ the quotient by this action, the set of isotopy classes of based $\mathbb{RP}^n$-structures:

$$\mathcal{D}_{\mathbb{RP}^n}(M) = \mathcal{D}'_{\mathbb{RP}^n}(M) / \mathrm{Diff}_0(M, m_0)$$

this set is endowed with the quotient topology. The group $PGL_n(\mathbb{R})$ acts on $\mathcal{D}'_{\mathbb{RP}^n}(M)$ by composition on f and by conjugation on ρ, and this action passes to the quotient $\mathcal{D}_{\mathbb{RP}^n}(M)$. We will denote the quotient by

$$\mathcal{T}_{\mathbb{RP}^n}(M) = \mathcal{D}_{\mathbb{RP}}(M) / PGL_n(\mathbb{R})$$

This set is endowed with the quotient topology. It is in natural bijection with the set of marked $\mathbb{RP}^n$-structures up to isotopy.

Let M be a closed orientable n-manifold such that its universal covering is $\mathbb{R}^n$ and the fundamental group $\pi_1(M)$ is Gromov hyperbolic. For example every closed orientable hyperbolic n-manifold satisfies the hypotheses. Note that if $n = 2$ or 3 only hyperbolic manifolds satisfy the hypotheses (this follows from the classification of surfaces and from Perelman's geometrization theorem).

We denote by $\mathcal{D}^c_{\mathbb{RP}^n}(M) \subset \mathcal{D}_{\mathbb{RP}^n}(M)$ and $\mathcal{T}^c_{\mathbb{RP}^n}(M) \subset \mathcal{T}_{\mathbb{RP}^n}(M)$ the subsets corresponding to convex projective structures on M, that are automatically strictly convex as $\pi_1(M)$ is Gromov hyperbolic. These subsets are open, by the Koskul openness theorem.

The holonomy map

$$\mathrm{hol}'_{\mathcal{D}} : \mathcal{D}'_{\mathbb{RP}}(M) \ni (f, \rho) \mapsto \rho \in \mathrm{Hom}(\pi_1(M, m_0), PGL_n(\mathbb{R}))$$

is continuous and it is invariant under the action of $\mathrm{Diff}_0(M, m_0)$, hence it defines a continuous map

$$\mathrm{hol}_{\mathcal{D}} : \mathcal{D}_{\mathbb{RP}}(M) \mapsto \mathrm{Hom}(\pi_1(M, m_0), PGL_n(\mathbb{R}))$$

The group $PGL_n(\mathbb{R})$ acts on $\mathrm{Hom}(\pi_1(M, m_0), PGL_n(\mathbb{R}))$ by conjugation, and on $\mathcal{D}_{\mathbb{RP}}(M)$ as said. The map $\mathrm{hol}_{\mathcal{D}}$ is equivariant with respect to these $PGL_n(\mathbb{R})$-actions, hence it induces a continuous map

$$\mathrm{hol}_{\mathcal{T}} : \mathcal{T}_{\mathbb{RP}^n}(M) \mapsto \mathrm{Hom}(\pi_1(M, m_0), PGL_n(\mathbb{R}))/PGL_n(\mathbb{R})$$

THEOREM 7.1. *The holonomy maps* $\mathrm{hol}_{\mathcal{D}}$ *and* $\mathrm{hol}_{\mathcal{T}}$*, when restricted to* $\mathcal{D}^c_{\mathbb{RP}^n}(M)$ *and* $\mathcal{T}^c_{\mathbb{RP}^n}(M)$ *respectively, are topological immersions, identifying these spaces with their images. These images are open subsets of* $\mathrm{Hom}(\pi_1(M), PGL_{n+1}(\mathbb{R}))$ *and* $\mathrm{Hom}(\pi_1(M), PGL_{n+1}(\mathbb{R}))/PGL_{n+1}(\mathbb{R})$ *respectively, and they contain only absolutely irreducible representations.*

PROOF. See [**3**]. The first part is based on [**10**], while the fact that the representations in the image are absolutely irreducible is based on [**6**]. $\square$

THEOREM 7.2. *The image of the map*

$$\pi_* : \mathrm{Hom}(\pi_1(M), SL^{\pm}_{n+1}(\mathbb{R})) \mapsto \mathrm{Hom}(\pi_1(M), PGL_{n+1}(\mathbb{R}))$$

contains the deformation space $\mathcal{D}^c_{\mathbb{RP}^n}(M)$*. This map has a canonical section, identifying* $\mathcal{D}^c_{\mathbb{RP}^n}(M)$ *with a finite union of connected components of the real algebraic set* $\mathrm{Hom}(\pi_1(M), SL_{n+1}(\mathbb{R}))$*. In particular* $\mathcal{D}^c_{\mathbb{RP}^n}(M)$ *is a closed semi-algebraic set.*

PROOF. See [**3**]. The fact that $\mathcal{D}^c_{\mathbb{RP}^n}(M)$ is closed follows from [**7**]. Note that as $\pi_1(M)$ is Gromov hyperbolic, then it is also virtually centerless. $\square$

THEOREM 7.3. *Consider the semi-geometric quotient (as in theorem 6.1)* $t : \mathrm{Hom}(\pi_1(M), SL_{n+1}(\mathbb{R})) \mapsto \overline{\mathrm{Char}}(\pi_1(M), SL_{n+1}(\mathbb{R}))$*. The image* $t(\mathcal{D}^c_{\mathbb{RP}^n}(M))$ *can be identified with the space* $\mathcal{T}^c_{\mathbb{RP}^n}(M)$*, it is a finite union of connected components (and, in particular, a clopen semi-algebraic subset) of* $\overline{\mathrm{Char}}(\pi_1(M), SL_{n+1}(\mathbb{R}))$*.*

PROOF. See [**3**]. $\square$

Now we present a result showing that the space $\mathcal{T}^c_{\mathbb{RP}^n}(M)$ is often big enough to be interesting, as there are cases where we know a lower bound on the dimension of this space.

PROPOSITION 7.4. *Suppose that* M *is a closed hyperbolic* n*-manifold containing* r *two-sided disjoint connected totally geodesic hypersurfaces. Then*

$$\dim \mathcal{T}^c_{\mathbb{RP}^n}(M) \geq r$$

Moreover for all n *is it possible to find such manifolds with arbitrarily large* r*.*

PROOF. See [**3**]. This result is based on [**9**]. $\square$

8. Compactification of semi-algebraic sets

8.1. Logarithmic limit sets. Let $V \subset (\mathbb{R}_{>0})^n$ be a real semi-algebraic set. We apply the **Maslov dequantization** to V: for $t \in (0, 1)$ the **amoeba** of V is

$$\mathcal{A}_t(V) = \{(\log_{(\frac{1}{t})}(x_1), \ldots, \log_{(\frac{1}{t})}(x_n)) \mid (x_1, \ldots, x_n) \in V\}$$

We can construct the deformation

$$W = \{(x,t) \in \mathbb{R}^n \times (0,\varepsilon) \mid x \in \mathcal{A}_t(V)\}$$

We denote by $\overline{W}$ the closure of W in $\mathbb{R}^n \times [0,\varepsilon)$, then we define

$$\mathcal{A}_0(V) = \pi(\overline{W} \cap \mathbb{R}^n \times \{0\}) \subset \mathbb{R}^n$$

where $\pi : \mathbb{R}^n \times [0,\varepsilon) \mapsto \mathbb{R}^n$ is the projection on the first factor. The set $\mathcal{A}_0(V)$ is the **logarithmic limit set** of V, the limit of the amoebas

THEOREM 8.1. *Let $V \subset (\mathbb{R}_{>0})^n$ be a semi-algebraic set. Then the logarithmic limit set $\mathcal{A}_0(V) \subset \mathbb{R}^n$ is a polyhedral cone, $\dim \mathcal{A}_0(V) \leq \dim V$ and $\mathcal{A}_0(V) \cap \mathbb{Q}^n$ is dense in $\mathcal{A}_0(V)$.*

PROOF. See [**2**]. □

Let $\mathbb{F}$ be a non-archimedean real closed field of rank one extending $\mathbb{R}$. The convex hull of $\mathbb{R}$ in $\mathbb{F}$ is a valuation ring denoted by $\mathcal{O}_{\leq}$. This valuation ring defines a valuation $v : \mathbb{F}^* \mapsto \mathbb{R}$. If $V \subset (\mathbb{F}_{>0})^n$ is a semi-algebraic set, the non-archimedean amoeba of V is defined as

$$\mathcal{A}(V) = \{(-v(x_1), \ldots, -v(x_n)) \mid (x_1, \ldots, x_n) \in V\}$$

THEOREM 8.2. *There exists a field $\mathbb{F}$ extending $\mathbb{R}$ that is real closed and non-archimedean of rank one such that for every semi-algebraic set $V \subset (\mathbb{R}_{>0})^n$, if $V^{\mathbb{F}}$ is the extension of V to $\mathbb{F}$, then:*

$$\mathcal{A}_0(V) = \mathcal{A}(V^{\mathbb{F}})$$

PROOF. See [**2**]. □

8.2. Compactification. If $V \subset (\mathbb{R}_{>0})^n$ is a closed semi-algebraic set, we can construct a compactification for V using its logarithmic limit set. $\mathcal{A}_0(V)$ represents the behavior at infinity of the amoeba, hence it can be used to compactify it. We take the quotient by the spherical equivalence relation

$$x \sim y \Leftrightarrow \exists \lambda > 0 : x = \lambda y$$

and we get the boundary

$$\partial V = (\mathcal{A}_0(V) \setminus \{0\})/\sim \; \subset S^{n-1}$$

Now we glue ∂V to V at infinity in the following way. We compactify $\mathbb{R}^n$ by adding the sphere at infinity:

$$\mathbb{R}^n \ni x \mapsto \frac{x}{\sqrt{1 + \|x\|^2}} \in D^n \qquad\qquad D^n \approx \mathbb{R}^n \cup S^{n-1}$$

Given a $t_0 < 1$, we will denote by $\overline{V}$ the closure of $\mathcal{A}_{t_0}(V)$ in D^n. Then

$$\overline{V} = \mathcal{A}_{t_0}(V) \cup \partial V$$

PROPOSITION 8.3. *The map $\mathrm{Log}_{\left(\frac{1}{t_0}\right)} : V \mapsto \overline{V}$ is a compactification of V. The compactification does not depend on the choice of t_0.*

PROOF. See [**3**]. □

Note that the logarithmic limit set $\mathcal{A}_0(V)$ is the cone over the boundary, and for this reason it will sometimes be denoted by $C(\partial V)$.

This construction can be generalized in a way that does not depend on the immersion of V in $\mathbb{R}^n$. Let $V \subset \mathbb{R}^n$ be a semi-algebraic set. A finite family of continuous semi-algebraic functions $\mathcal{F} = \{f_1, \dots, f_m\}$, with $f_i : V \mapsto \mathbb{R}_{>0}$, is called a **proper family** if the map

$$E_{\mathcal{F}} : V \ni x \mapsto (f_1(x), \dots, f_m(x)) \in (\mathbb{R}_{>0})^m$$

is proper. In this case the map $L_{\mathcal{F}} = \mathrm{Log}_{\left(\frac{1}{t_0}\right)} \circ E_{\mathcal{F}}$ is also proper.

The image $E_{\mathcal{F}}(V) \subset (\mathbb{R}_{>0})^n$ is a closed semi-algebraic subset, and we can compactify it as before, by $\overline{E_{\mathcal{F}}(V)} = \mathcal{A}_{t_0}(E_{\mathcal{F}}(V)) \cup \partial E_{\mathcal{F}}(V)$.

Let $\hat{V} = V \cup \{\infty\}$ denote the Alexandrov compactification of V. Consider the map

$$i : V \ni x \mapsto (x, L_{\mathcal{F}}(x)) \in \hat{V} \times \overline{E_{\mathcal{F}}(V)}$$

and let $\overline{V}_{\mathcal{F}}$ be the closure of the image $i(V)$ in $\hat{V} \times \overline{E_{\mathcal{F}}(V)}$.

Proposition 8.4. *The map $i : V \mapsto \overline{V}_{\mathcal{F}}$ is a compactification of V. The boundary $\partial_{\mathcal{F}} V = \overline{V}_{\mathcal{F}} \setminus i(V)$ is the set $\partial E_{\mathcal{F}}(V)$.*

Proof. See [**3**]. □

The cone over the boundary will be denoted by $C(\partial_{\mathcal{F}} V) = \mathcal{A}_0(E_{\mathcal{F}}(V))$.

A further generalization of the construction of the compactification is needed if we want to extend the action of a group on the semi-algebraic set to an action on the compactification, as in subsection 8.3.

Let $V \subset \mathbb{R}^n$ be a semi-algebraic set. A (possibly infinite) family of continuous semi-algebraic functions $\mathcal{G} = \{f_i\}_{i \in I}$, with $f_i : V \mapsto \mathbb{R}_{>0}$, is called a **proper family** if there exist a finite subfamily $\mathcal{F} \subset \mathcal{G}$ that is proper.

Suppose that $\mathcal{G}$ is proper. Let

$$P_{\mathcal{G}} = \{\mathcal{F} \subset \mathcal{G} \mid \mathcal{F} \text{ is proper } \}$$

a non-empty set partially ordered by inclusion. If $\mathcal{F} \subset \mathcal{F}'$ we denote by $\pi_{\mathcal{F}',\mathcal{F}}$ the projection

$$\pi_{\mathcal{F}',\mathcal{F}} : \mathbb{R}^{\mathcal{F}'} \mapsto \mathbb{R}^{\mathcal{F}}$$

on the coordinates corresponding to $\mathcal{F}$. This projection restricts to a surjective map

$$\pi_{\mathcal{F}',\mathcal{F}|\mathcal{A}_{t_0}(E_{\mathcal{F}'}(V))} : \mathcal{A}_{t_0}(E_{\mathcal{F}'}(V)) \mapsto \mathcal{A}_{t_0}(E_{\mathcal{F}}(V))$$

By [**1**, prop. 4.7], the restriction to the logarithmic limit sets is also surjective:

$$\pi_{\mathcal{F}',\mathcal{F}|\mathcal{A}_0(E_{\mathcal{F}'}(V))} : \mathcal{A}_0(E_{\mathcal{F}'}(V)) \mapsto \mathcal{A}_0(E_{\mathcal{F}}(V))$$

Proposition 8.5. *Let $\mathcal{F}, \mathcal{F}' \in P_{\mathcal{G}}$. If $\mathcal{F} \subset \mathcal{F}'$, the map $\pi_{\mathcal{F}',\mathcal{F}|\mathcal{A}_0(E_{\mathcal{F}'}(V))}$ induces a map*

$$\partial\pi_{\mathcal{F}',\mathcal{F}} : \partial_{\mathcal{F}'} V \mapsto \partial_{\mathcal{F}} V$$

Proof. See [**3**]. □

The maps $\pi_{\mathcal{F}',\mathcal{F}}$ and $\partial\pi_{\mathcal{F}',\mathcal{F}}$ define three inverse systems:

$$\{\mathcal{A}_{t_0}(E_{\mathcal{F}}(V))\}_{\mathcal{F} \in P_{\mathcal{G}}}, \{\mathcal{A}_0(E_{\mathcal{F}}(V))\}_{\mathcal{F} \in P_{\mathcal{G}}}, \{\partial_{\mathcal{F}} V\}_{\mathcal{F} \in P_{\mathcal{G}}}$$

Consider the inverse limit

$$L = \varprojlim \mathcal{A}_{t_0}(E_{\mathcal{F}}(V))$$

we will denote by $\pi_{\mathcal{G},\mathcal{F}} : L \mapsto \mathcal{A}_{t_0}(E_{\mathcal{F}}(V))$ the canonical projection. By the explicit description of the inverse limit, L is a closed subset of the product:

$$\left\{ (x_{\mathcal{F}}) \in \prod_{\mathcal{F}\in P_{\mathcal{G}}} \mathcal{A}_{t_0}(E_{\mathcal{F}}(V)) \mid \forall \mathcal{F} \subset \mathcal{F}', \pi_{\mathcal{F}',\mathcal{F}}(x_{\mathcal{F}'}) = x_{\mathcal{F}} \right\}$$

For every $x \in L$, and every $f \in \mathcal{G}$, let $\mathcal{F}$ be a proper finite family containing f. Then the value of the f-coordinate of the point $\pi_{\mathcal{G},\mathcal{F}}(x)$ does not depend on the choice of the family $\mathcal{F}$. This value will be denoted by x_f. The map

$$L \ni x \mapsto (x_f)_{f\in\mathcal{G}} \in \mathbb{R}^{\mathcal{G}}$$

identifies L with a subset of $\mathbb{R}^{\mathcal{G}}$.

The system of maps $L_{\mathcal{F}} : V \mapsto \mathcal{A}_{t_0}(E_{\mathcal{F}}(V))$, defined for every $\mathcal{F} \in P_{\mathcal{G}}$, induces by the universal property a well defined map $L_{\mathcal{G}} : V \mapsto L$.

Proposition 8.6. *The map $L_{\mathcal{G}}$ is surjective and proper, and it can be identified with the map*

$$V \ni x \mapsto \left(\log_{\left(\frac{1}{t_0}\right)}(f(x)) \right)_{f\in\mathcal{G}} \in \mathbb{R}^{\mathcal{G}}$$

Proof. See [**3**]. □

As the map $L_{\mathcal{G}}$ is surjective, in the following we will denote L by $L_{\mathcal{G}}(V)$. Now consider the inverse limit

$$M = \varprojlim \overline{E_{\mathcal{F}}(V)} = \varprojlim \mathcal{A}_{t_0}(E_{\mathcal{F}}(V)) \cup \partial_{\mathcal{F}} V$$

The space M is compact, as it is an inverse limit of compact spaces, and we will use the map $L_{\mathcal{G}} : V \mapsto M$ to define a compactification, as in the previous subsection.

Consider the map

$$i : V \ni x \mapsto (x, L_{\mathcal{G}}(x)) \in \hat{V} \times M$$

Let $\overline{V}_{\mathcal{G}}$ be the closure of the image $i(V)$ in $\hat{V} \times M$.

Proposition 8.7. *The map $i : V \mapsto \overline{V}_{\mathcal{G}}$ is a compactification of V. The boundary $\partial_{\mathcal{G}} V = \overline{V}_{\mathcal{G}} \setminus i(V)$ is the set $\varprojlim \partial_{\mathcal{F}} V$.*

Proof. See [**3**]. □

The limit $\partial_{\mathcal{G}} V$ is the spherical quotient of the limit

$$C(\partial_{\mathcal{G}} V) = \varprojlim C(\partial_{\mathcal{F}} V) = \varprojlim \mathcal{A}_0(E_{\mathcal{F}}(V))$$

More explicitly, $C(\partial_{\mathcal{G}} V)$ is a closed subset of the product:

$$\left\{ (x_{\mathcal{F}}) \in \prod_{\mathcal{F}\in P_{\mathcal{G}}} \mathcal{A}_0(E_{\mathcal{F}}(V)) \mid \forall \mathcal{F} \subset \mathcal{F}', \pi_{\mathcal{F}',\mathcal{F}}(x_{\mathcal{F}'}) = x_{\mathcal{F}} \right\}$$

As before, for every $x \in C(\partial_{\mathcal{G}} V)$, and every $f \in \mathcal{G}$, let $\mathcal{F}$ be a proper finite family containing f. Then the value of the f-coordinate of the point $\pi_{\mathcal{G},\mathcal{F}}(x)$ does not depend on the choice of the family $\mathcal{F}$. This value will be denoted by x_f. The map

$$C(\partial_{\mathcal{G}} V) \ni x \mapsto (x_f)_{f \in \mathcal{G}} \in \mathbb{R}^{\mathcal{G}}$$

identifies $C(\partial_{\mathcal{G}} V)$ with a closed subset of $\mathbb{R}^{\mathcal{G}}$.

8.3. Group actions. Let G be a group acting with continuous semi-algebraic maps on a semi-algebraic set $V \subset \mathbb{R}^n$. Suppose that $\mathcal{G}$ is a (possibly infinite) proper family of functions $V \mapsto \mathbb{R}_{>0}$, and that $\mathcal{G}$ is invariant for the action of G.

Then the action of G on V extends continuously to an action on the compactification $\overline{V}_{\mathcal{G}}$.

As $\mathcal{G}$ is invariant for the action of G, if we see the limits $L_{\mathcal{G}}(V)$ and $C(\partial_{\mathcal{G}} V)$ as subsets of $\mathbb{R}^{\mathcal{G}}$, then G acts on $L_{\mathcal{G}}(V)$ and $C(\partial_{\mathcal{G}} V)$ by a permutation of the coordinates corresponding to the action on $\mathcal{G}$, and this action induces an action on the spherical quotient of $C(\partial_{\mathcal{G}} V)$, the boundary $\partial_{\mathcal{G}}$.

Note that the map $L_{\mathcal{G}} : V \mapsto L_{\mathcal{G}}(V)$ is equivariant for this action, hence the action of G on $\partial_{\mathcal{G}}$ extends continuously the action of G on V.

8.4. Non-archimedean description. Let $V \subset \mathbb{R}^n$ be a semi-algebraic set, and let $\mathcal{G}$ be a (possibly infinite) proper family of continuous semi-algebraic functions $V \mapsto \mathbb{R}_{>0}$.

Let $\mathbb{F}$ be a real closed non-archimedean field with finite rank extending $\mathbb{R}$. The convex hull of $\mathbb{R}$ in $\mathbb{F}$ is a valuation ring denoted by $\mathcal{O}_{\leq}$. This valuation ring defines a valuation $v : \mathbb{F}^* \mapsto \Lambda$, where Λ is an ordered abelian group. As $\mathbb{F}$ has finite rank, the group Λ has only finitely many convex subgroups $0 = \Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_r = \Lambda$. The number r of convex subgroups is the **rank** of the field $\mathbb{F}$.

The quotient $\Lambda \mapsto \Lambda/\Lambda_{r-1}$ is an ordered group of rank one, hence it is isomorphic to a subgroup of $\mathbb{R}$. We fix one of these isomorphisms, and we denote by $\overline{v}$ the composition of the valuation v with the quotient map $\Lambda \mapsto \Lambda/\Lambda_{r-1}$, another valuation of $\mathbb{F}$ that is real valued:

$$\overline{v} : \mathbb{F}^* \mapsto \mathbb{R}$$

Let $V^{\mathbb{F}}$ be the extension of V to $\mathbb{F}$, a semi-algebraic subset of $(\mathbb{F}_{>0})^n$. Let $\mathcal{G}^{\mathbb{F}} = \{f^{\mathbb{F}} \mid f \in \mathcal{G}\}$, where $f^{\mathbb{F}} : V^{\mathbb{F}} \mapsto \mathbb{F}_{>0}$ is the extension of the function $f : V \mapsto \mathbb{R}_{>0}$.

Let $\mathcal{F} = \{f_1, \ldots, f_m\} \subset \mathcal{G}$ be a finite proper family. We denote the corresponding family of extensions by $\mathcal{F}^{\mathbb{F}} = \{f_1^{\mathbb{F}}, \ldots, f_m^{\mathbb{F}}\} \subset \mathcal{G}^{\mathbb{F}}$, and we will denote by $E_{\mathcal{F}}^{\mathbb{F}} : V^{\mathbb{F}} \mapsto (\mathbb{F}_{>0})^m$ the extension of the map $E_{\mathcal{F}}$.

PROPOSITION 8.8. *The image of the map*

$$\operatorname{Log} : (\mathbb{F}_{>0})^n \supset E_{\mathcal{F}}^{\mathbb{F}}(V^{\mathbb{F}}) \ni x \mapsto (-\overline{v}(x_1), \ldots, -\overline{v}(x_n)) \in \mathbb{R}^m$$

is contained in the logarithmic limit set $\mathcal{A}_0(E_{\mathcal{F}}(V))$.

PROOF. See [**3**]. □

In other words, the image of the map

$$\operatorname{Log}_{\mathcal{F}} = \operatorname{Log} \circ E_{\mathcal{F}}^{\mathbb{F}} : V^{\mathbb{F}} \ni x \mapsto (-\overline{v}(f_1(x)), \ldots, -v(f_m(x))) \in \mathbb{R}^m$$

is contained in $\mathcal{A}_0(E_{\mathcal{F}}(V)) = C(\partial_{\mathcal{F}} V)$.

The system of maps $\mathrm{Log}_{\mathcal{F}} : \overline{V} \mapsto C(\partial_{\mathcal{F}} V)$, defined for every $\mathcal{F} \in P_{\mathcal{G}}$, induces by the universal property a well defined map $\mathrm{Log}_{\mathcal{G}} : V \mapsto C(\partial_{\mathcal{G}} V)$. The map $\mathrm{Log}_{\mathcal{G}}$ can be identified with the map

$$\overline{V} \ni x \mapsto \left(-\overline{v}(f(x))\right)_{f\in\mathcal{G}} \in C(\partial_{\mathcal{G}} V) \subset \mathbb{R}^{\mathcal{G}}$$

THEOREM 8.9. *Let $V \subset \mathbb{R}^n$ be a semi-algebraic set, and let $\mathcal{G}$ be a proper family of positive continuous semi-algebraic functions on V. There exists a real closed non-archimedean field $\mathbb{F}$ with finite rank extending $\mathbb{R}$ such that if $V^{\mathbb{F}}$ is the extension of V to the field $\mathbb{F}$, then $\mathrm{Log}_{\mathcal{G}}(V^{\mathbb{F}}) = C(\partial_{\mathcal{G}} V)$.*

PROOF. See [**3**]. □

9. Degeneration of projective structures

9.1. Compactification of the parameter space. Let M be a closed orientable n-manifold such that the universal covering is $\mathbb{R}^n$ and the fundamental group $\pi_1(M)$ is Gromov hyperbolic. We want to construct a compactification of the space $\mathcal{T}^c_{\mathbb{RP}^n}(M)$ of marked convex projective structures on M, using the structure of semi-algebraic set it inherits from its identification with a connected component of $\overline{\mathrm{Char}}(\pi_1(M), SL_n(\mathbb{R}))$.

For every element $p \in \mathcal{T}^c_{\mathbb{RP}^n}(M)$ and $\gamma \in \pi_1(M)$, we recall that $\ell_\gamma(p)$ is translation length of γ for the structure p, and we denote by $e_\gamma(p)$ the ratio $\frac{\lambda_1}{\lambda_{n+1}}$ between the eigenvalues of maximum and minimum modulus of the conjugacy class of matrices $p(\gamma)$. Then the function

$$e_\gamma : \mathcal{T}^c_{\mathbb{RP}^n}(M) \mapsto \mathbb{R}_{>0}$$

is a semi-algebraic function on $\mathcal{T}^c_{\mathbb{RP}^n}(M)$, such that $\log_e(e_\gamma(p)) = \ell_\gamma(p)$.

Let $\mathcal{G} = \{e_\gamma\}_{\gamma\in\pi_1(M)}$.

PROPOSITION 9.1. *There exist a finite subset $A \subset \mathcal{G}$ such that the family $\mathcal{F}_A = \{e_\gamma\}_{\gamma\in A}$ is proper.*

PROOF. See [**3**]. □

As the family $\mathcal{G}$ is a proper family, it defines a compactification

$$\overline{\mathcal{T}^c_{\mathbb{RP}^n}(M)}_{\mathcal{G}} = \mathcal{T}^c_{\mathbb{RP}^n}(M) \cup \partial_{\mathcal{G}} \mathcal{T}^c_{\mathbb{RP}^n}(M)$$

As the family $\mathcal{G}$ is invariant for the action of the mapping class group of M, the action of the mapping class group extends continuously to an action on $\overline{\mathcal{T}^c_{\mathbb{RP}^n}(M)}_{\mathcal{G}}$.

Note that this compactification is constructed taking the limits of the functions $\log_e \circ e_\gamma$, i.e. the translation length functions ℓ_γ.

9.2. Interpretation of the boundary points. Now we investigate which objects can be used for the interpretation of the boundary points. A point in the parameter space $\mathcal{T}^c_{\mathbb{RP}^n}(M)$ corresponds to a marked convex projective structure on M. In other words it corresponds to a conjugacy class of development pairs (D, h), where $h : \pi_1(M) \mapsto SL_{n+1}(\mathbb{R})$, and $D : \widetilde{M} \mapsto \mathbb{RP}^n$ is an h-equivariant map. We want to extend this interpretation to the boundary points. We will associate with every boundary points a class of tropical projective structures on M, were two tropical projective structures corresponds to the same boundary point if and only if they have the same marked length spectrum. In other words a boundary point is interpreted as a marked length spectrum of a tropical projective structure on M.

Here we give a geometric interpretation to the points of the boundaries of the spaces of convex projective structures. Every action of $\pi_1(M)$ on a tropical projective space $FP^n(\mathbb{F})$ has a well defined length spectrum $(l(\gamma))_{\gamma\in\pi_1(M)} \in \mathbb{R}^{\pi_1(M)}$.

THEOREM 9.2. *Let $\mathbb{F}$ be a field as in theorem 8.9. The points of $C(\partial_{\mathcal{G}}\mathcal{T}^c_{\mathbb{RP}^n}(M))$ are marked length spectra of tropical projective structures on M, constructed using the tropical projective space $FP^n(\mathbb{F})$.*

PROOF. Let $\mathcal{T}^c_{\mathbb{RP}^n}(M)^{\mathbb{F}} \subset \overline{\mathrm{Char}}(\pi_1(M), SL_{n+1}(\mathbb{F}))$ be the extension of the real semi-algebraic set $\mathcal{T}^c_{\mathbb{RP}^n}(M)$ to the field $\mathbb{F}$. Every element of $\mathcal{T}^c_{\mathbb{RP}^n}(M)^{\mathbb{F}}$ is a conjugacy class of a representation $\rho : \pi_1(M) \mapsto SL_{n+1}(\mathbb{F})$.

Let $x \in C(\partial_{\mathcal{G}}\mathcal{T}^c_{\mathbb{RP}^n}(M)) \subset \mathbb{R}^{\mathcal{G}}$. As we said by theorem 8.9, there exists a representation $\rho \in \overline{\mathcal{T}^c_{\mathbb{RP}^n}(M)}$ such that for every $\gamma \in \pi_1(M)$, the matrix $\rho(\gamma)$ satisfies $x_{e_\gamma} = \tau\left(\left|\frac{\lambda_1}{\lambda_{n+1}}\right|\right)$.

By theorem 5.3 there exists a ρ-equivariant map $f : \widetilde{M} \mapsto FP^n(\mathbb{F})$. The pair (f, ρ) is a tropical projective structure on M. Consider the action of $\pi_1(M)$ on $FP^n(\mathbb{F})$ induced by the representation ρ. As we said above, the translation length of an element γ is

$$l(\rho(\gamma)) = \tau\left(\left|\frac{\lambda_1}{\lambda_{n+1}}\right|\right)$$

Hence the marked length spectrum of the tropical projective structure (f, ρ) identifies the boundary point. $\square$

The tropical projective structures on M should correspond to some "more standard" geometric structure on M. The situation we have here is very similar to what happens in the work of Morgan and Shalen. In their work S is a hyperbolic surface with an action of $\pi_1(S)$ on $FP^1(\mathbb{F})$. This case is well understood: FP^1 is a real tree and the equivariant map induces a measured lamination on S, that is "dual" to the action of $\pi_1(S)$ on the real tree.

This work can possibly lead to the discovery of analogous structures for the general case. For example an action of $\pi_1(M)$ on FP^m induces a degenerate metric on M, and this metric can be used to associate a length with each curve. Anyway it is not clear up to now how to classify these induced structures. This is closely related to a problem raised by J. Roberts (see [**14**, problem 12.19]): how to extend the theory of measured laminations to higher rank groups, such as, for example, $SL_{n+1}(\mathbb{R})$.

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